

4 Interim Report 6

3 A TREATMENT OF THE ^{MU}_u-BIAS 4

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A TREATMENT OF THE μ -BIAS

1. Introduction

Orbit determination schemes which do not account for the effect of earth model errors will usually furnish an optimistic covariance matrix. This leads then to residuals which are far outside their predicted range, thus severely limiting the length of arc over which the determination is valid. One of the important model errors arises from the uncertainty of the earth's gravitational constant μ . The treatment of this uncertainty in the following analysis uses μ as an additional state variable without, however, necessitating its redetermination or the re-evaluation of its variance.

2. Analysis

The inclusion of μ as a seventh state variable using the 6 α variables described in reference [1] [Modification of MINVAR program] will necessitate several changes.

a. State covariance matrix.

The Q matrix will now be a 7 x 7 matrix

$$Q_7 = \begin{bmatrix} Q_6 & C_\mu \\ C_\mu^* & \sigma_\mu^2 \end{bmatrix} \quad (1)$$

where Q_6 is the old (6 x 6) Q matrix and C_μ (a [6 x 1] matrix) is the cross correlation between α and μ . It is initially zero.

C_{μ}^* is its transpose and σ_{μ}^2 is the variance of μ about its mean value.

b. Updating the State Covariance Matrix

The new state covariance matrix is updated in time according to the formula:

$$Q_7(t) = \Omega_7(t, t_0) Q_7(t_0) \Omega_7^*(t, t_0).$$

The matrix Ω_7 is given by:

$$\Omega_7(t, t_0) = \begin{bmatrix} \Omega_6(t, t_0) & W(t, t_0) \\ 0 & I \end{bmatrix} \quad (2)$$

Ω_6 is the previously obtained state propagation matrix.

W is a (6 x 1) matrix obtained by differentiating the $\alpha(t)$'s with respect to $\mu(t_0)$. [see Appendix]

$$\begin{aligned} W_1 &= W_2 = W_5 = 0 \\ W_3 &= \frac{\partial \alpha_3(t)}{\partial \mu(t_0)} = \frac{h}{2v^2 r^3} \left(\Delta t + \frac{d_0}{\mu} \beta^2 F_2 \right) \\ &\quad + \frac{\dot{f} d_0}{2\mu v^2 h} \left[\frac{\mu}{r} g - d_0 + \frac{r_0 \dot{f} h^2}{\mu} \right] \end{aligned} \quad (3)$$

$$W_4 = \frac{\partial d(t)}{\partial \mu(t_0)} = \frac{1}{2} \frac{d}{\mu} - \frac{1}{2} \frac{d_0}{\mu} g + \frac{1}{2} \frac{\Delta t}{r} \left(\frac{rv^2}{\mu} - 1 \right) \quad (4)$$

$$W_6 = \frac{\partial r(t)}{\partial \mu(t_0)} = \frac{1}{2\mu r} \left(d \Delta t - d_0 g \right) \quad (5)$$

This leads to the following update formulas:

$$Q_6(t) = \Omega_6 Q_6(t_0) \Omega_6^* + \Omega_6 C_\mu(t_0) W^* + W C_\mu^*(t_0) \Omega_6^* + W \sigma_\mu^2 W^* \quad (6)$$

$$C_\mu(t) = \Omega_6 C_\mu + W \sigma_\mu^2 \quad (7)$$

$$\sigma_\mu^2(t) = \sigma_\mu^2(t_0) \quad (8)$$

c. State Transformation Matrix

Since the equations connecting the Cartesian state and the α -state involve μ , a change in μ must cause a change in the Cartesian state if all the α 's are kept constant. The Cartesian state is described by the (7 x 1) matrix

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} R \\ \dot{R} \\ \mu \end{bmatrix}$$

The α state is described by

$$\begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_6 \\ \alpha_7 \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \alpha_6 \\ \mu \end{bmatrix}$$

The state transformation matrix is given by:

$$S_7 = \frac{\partial x_i(t)}{\partial \alpha_j(t)}$$

This may be partitioned in the following manner:

$$S_7 = \begin{bmatrix} S_6 & T \\ 0 & I \end{bmatrix} \quad (9)$$

where T is a (6×1) matrix described by

$$T = \begin{bmatrix} \frac{\partial R}{\partial \mu} \\ \frac{\partial \dot{R}}{\partial \mu} \end{bmatrix} \quad (10)$$

Similarly, the inverse state transformation matrix will be a (7x7) matrix

$$S_7^{-1} = \frac{\partial \alpha_i}{\partial x_j} \quad \text{or} \quad S_7^{-1} = \begin{bmatrix} S_6^{-1} & V \\ 0 & I \end{bmatrix} \quad (11)$$

where V is a (6x1) matrix described by

$$V = \frac{\partial \alpha_i}{\partial \mu} \quad i = 1, 2, \dots, 6$$

Since $S_7^{-1} S_7$ must equal I , and the matrix product of the partitioned matrices gives

$$S_7^{-1} S_7 = \begin{bmatrix} S_6^{-1} S_6 & S_6^{-1} T + V \\ 0 & I \end{bmatrix},$$

then

$$S_6^{-1} T + V = 0$$

$$\text{or} \quad T = -S_6 V \quad (12)$$

The determination of V is relatively simple since none of the α 's contain μ explicitly except $\alpha_5 = \frac{2}{r} - \frac{v^2}{\mu}$. So

$$V = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{v^2}{\mu} \\ 0 \end{bmatrix} \quad (13)$$

from which one can determine

$$T = -\frac{v^2}{\mu} \quad [\text{fifth column of } S_6 \text{ matrix}]$$

$$T = \begin{bmatrix} -\frac{d}{2\mu h^2} & H \times R \\ \frac{1}{2\mu} & \dot{R} \end{bmatrix} \quad (14)$$

d. The partials of the observations with respect to the 7 state variables

$$\begin{aligned}
 N_7 &= \frac{\partial y}{\partial \alpha_{1-7}} \\
 &= \begin{bmatrix} \frac{\partial y}{\partial x_{1-6}} & \vdots & \frac{\partial y}{\partial \mu} \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial \alpha} & \frac{\partial x}{\partial \mu} \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} M & \vdots & 0 \end{bmatrix} \begin{bmatrix} S_6 & T \\ 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} M S_6 & \vdots & M T \end{bmatrix}
 \end{aligned}$$

but $T = -S_6 V$

and $N_6 = M S_6$

$$N_7 = \begin{bmatrix} N_6 & \vdots & -(N_6 V) \end{bmatrix} \quad (15)$$

The covariance matrix of the observation,

$$Y = N_7 Q_7 N_7^* + R,$$

becomes

$$Y = N Q N^* + R - N V C_\mu^* N^* - N C_\mu V^* N^* + N V \sigma_\mu^2 V^* N^* \quad (16)$$

At an observation, then,

$$\Delta \alpha = [Q N^* - C_u(V^* N^*)] Y^{-1} \Delta y \quad (17)$$

$$\begin{aligned} \Delta Q_6 = & Q N^* Y^{-1} N Q - Q N^* Y^{-1} N V C_\mu^* - C_\mu V^* N^* Y^{-1} N Q \\ & + C_\mu (V^* N^* Y^{-1} N V) C_\mu^* \end{aligned} \quad (18)$$

$$\Delta C_\mu = [Q N^* - C_\mu(V^* N^*)] Y^{-1} (N C_\mu - N V \sigma_\mu^2) \quad (19)$$

where N is the former N_6 and Q is Q_6 .

T could be derived directly by taking into account the following considerations:

1) α_1 and α_2 must not change as μ changes, i.e. $\frac{\partial R}{\partial \mu}$ and $\frac{\partial \dot{R}}{\partial \mu}$ must be in the plane defined by R and \dot{R} .

2) α_3 must not change, hence $\frac{\partial \dot{R}}{\partial \mu}$ must be parallel to \dot{R} .

Consequently:

$$T = \begin{bmatrix} \frac{\partial R}{\partial \mu} \\ \frac{\partial \dot{R}}{\partial \mu} \end{bmatrix} = \begin{bmatrix} K_1 R + K_2 \dot{R} \\ K_3 \dot{R} \end{bmatrix} \quad (20)$$

$$3) \quad \alpha_4 \cdot d = R \cdot \dot{R}$$

$$\alpha_5 = \frac{1}{a} = \frac{2}{(R \cdot R)^{\frac{1}{2}}} = \frac{\dot{R} \cdot \dot{R}}{\mu} \quad (21)$$

$$\alpha_6 = r \cdot (R \cdot R)^{\frac{1}{2}}$$

must not change.

Hence

$$R \cdot \frac{\partial \dot{R}}{\partial \mu} + \frac{\partial R}{\partial \mu} \cdot \dot{R} = 0$$

$$R \cdot \frac{\partial R}{\partial \mu} = 0 \quad (22)$$

$$\frac{-2 \dot{R} \cdot \frac{\partial \dot{R}}{\partial \mu}}{\mu} + \frac{\dot{R} \cdot \dot{R}}{\mu^2} = 0$$

Substituting in the expressions for $\frac{\partial R}{\partial \mu}$ and $\frac{\partial \dot{R}}{\partial \mu}$ from equation (20) gives

$$R \cdot [K_3 \dot{R}] + \dot{R} \cdot [K_1 R + K_2 \dot{R}] = 0$$

$$R \cdot [K_1 R + K_2 \dot{R}] = 0$$

$$-\frac{2 \dot{R}}{\mu} [K_3 \dot{R}] + \frac{\dot{R} \cdot \dot{R}}{\mu^2} = 0$$

$$K_1 d + K_2 v^2 + K_3 d = 0$$

$$K_1 r^2 + K_2 d = 0$$

$$-\frac{2 K_3 v^2}{\mu} = -\frac{v^2}{\mu^2}$$

$$K_3 = \frac{1}{2\mu} \quad (23)$$

$$\begin{array}{l|l} K_1 d + K_2 v^2 = -\frac{d}{2\mu} & -d \\ K_1 r^2 + K_2 d = 0 & v^2 \end{array}$$

$$K_1 (r^2 v^2 - d^2) = \frac{d^2}{2\mu}$$

$$K_1 = \frac{d^2}{2\mu h^2} \quad (24)$$

$$K_2 = -\frac{d^2 r^2}{2\mu h^2 d} = -\frac{d r^2}{2\mu h^2} \quad (25)$$

Putting these K coefficients back into equation (20) gives

$$\begin{aligned} \frac{\partial R}{\partial \mu} &= -\frac{d}{2\mu h^2} \left(r^2 \dot{R} - d R \right) = -\frac{d}{2\mu h^2} H X R \\ \frac{\partial \dot{R}}{\partial \mu} &= \frac{1}{2\mu} \dot{R} \end{aligned} \quad (26)$$

which is the same T as previously determined.

APPENDIX

Derivation of the W Matrix

The W matrix is a (6 x 1) matrix defined by

$$W = \frac{\partial \alpha_i(t)}{\partial \mu(t_0)}$$

$$W_1 = W_2 = W_5 = 0$$

We need $\frac{\partial t}{\partial \mu}$ from Kepler's equation

$$\sqrt{\mu} \Delta t = \beta^3 F_1 + r_0 \beta F_3 + \frac{d_0}{\sqrt{\mu}} \beta^2 F_2$$

$$\sqrt{\mu} \Delta t = a^{3/2} (\theta - \sin \theta) + r_0 \sqrt{a} \sin \theta + \frac{d_0}{\sqrt{\mu}} a (1 - \cos \theta)$$

Differentiating w. r. t. $\mu(t_0)$

$$\frac{1}{2} \frac{\Delta t}{\sqrt{\mu}} = \sqrt{a} r \frac{\partial \theta}{\partial \mu(t_0)} - \frac{1}{2} \frac{d_0}{\mu^{3/2}} a (1 - \cos \theta)$$

$$\frac{\partial \theta}{\partial \mu(t_0)} = \frac{1}{2} \frac{\Delta t}{\sqrt{\mu} a r} + \frac{1}{2 \sqrt{a} r} \frac{d_0}{\mu^{3/2}} \beta^2 F_2$$

$$\alpha_6(t) = r \beta^2 F_2 + r_0 F_4 + \frac{d_0}{\sqrt{\mu}} \beta F_3$$

$$= a (1 - \cos \theta) + r_0 \cos \theta + \frac{d_0}{\sqrt{\mu}} \sqrt{a} \sin \theta$$

$$\begin{aligned}
W_6 &= \frac{\partial \alpha_6(t)}{\partial \mu(t_0)} = \frac{\partial r}{\partial \theta} \frac{\partial \theta}{\partial \mu(t_0)} - \frac{1}{2} \frac{d_0}{\mu^{3/2}} \beta F_3 \\
&= \frac{\sqrt{a} d}{\sqrt{\mu}} \left(\frac{1}{2} \frac{\Delta t}{\sqrt{\mu a} r} + \frac{1}{2 \sqrt{a} r} \frac{d_0}{\mu^{3/2}} \beta^2 F_2 \right) - \frac{1}{2} \frac{d_0}{\mu^{3/2}} \beta F_3 \\
&= \frac{1}{2 \mu r} (d \Delta t - d_0 g)
\end{aligned}$$

$$\begin{aligned}
\alpha_1(t) &= d + \sqrt{\mu} \left(1 - \frac{r_0}{a} \right) \beta F_3 + d_0 F_4 \\
&= \sqrt{\mu} \left(1 - \frac{r_0}{a} \right) \sqrt{a} \sin \theta + d_0 \cos \theta
\end{aligned}$$

$$= d_0 + \sqrt{\mu a} \theta - \frac{\mu}{a} \Delta t$$

$$\begin{aligned}
W &= \frac{\partial \alpha_4(t)}{\partial \mu(t_0)} = \frac{\partial d}{\partial \mu(t_0)} + \sqrt{\mu a} \frac{\partial \theta}{\partial \mu(t_0)} + \frac{\sqrt{a} \theta}{\sqrt{\mu}} - \frac{1}{a} \Delta t \\
&= \frac{1}{2} \frac{\Delta t}{r} + \frac{1}{2 r} \frac{d_0}{\mu} \beta^2 F_2 + \frac{\beta}{2 \sqrt{\mu}} - \frac{1}{a} \Delta t
\end{aligned}$$

$$\frac{1}{2 \mu} = \frac{1}{2} \frac{d}{\mu} - \frac{1}{2} \frac{d_0}{\mu} + \frac{1}{2} \frac{\Delta t}{a}$$

$$W_4 = \frac{1}{2} \frac{d}{\mu} - \frac{1}{2} \frac{d_0}{\mu} \left(1 - \frac{1}{r} \beta^2 F_2 \right) + \frac{1}{2} \frac{\Delta t}{r} \left(1 - \frac{r}{a} \right)$$

$$W_4 = \frac{1}{2} \frac{d}{\mu} - \frac{1}{2} \frac{d_0}{\mu} g + \frac{1}{2} \frac{\Delta t}{r} \left(\frac{r v^2}{\mu} - 1 \right)$$

In order to obtain W_3 there is the general expression

$$\frac{\partial \alpha_3(t)}{\partial \alpha_j(0)} = \frac{1}{h v^2} H X \dot{R} \cdot \left(\dot{f} \frac{\partial R_0}{\partial \alpha_j(0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \alpha_j(0)} \right) + \frac{h}{v^2} \left(\dot{f} \frac{\partial \dot{g}}{\partial \alpha_j(0)} - \dot{g} \frac{\partial \dot{f}}{\partial \alpha_j(0)} \right)$$

The expressions $\frac{\partial R_0}{\partial \mu(t_0)}$ and $\frac{\partial \dot{R}_0}{\partial \mu(t_0)}$ come from the seventh column of the S_7 matrix evaluated at t_0 . So

$$\begin{aligned} -\frac{1}{h v^2} H X \dot{R} \cdot \left(\dot{f} \frac{\partial R_0}{\partial \mu(t_0)} + \dot{g} \frac{\partial \dot{R}_0}{\partial \mu(t_0)} \right) &= \\ \frac{1}{2 \mu h v^2} \left[(H X \dot{R}) \cdot (H X R_0) \left(\frac{-\dot{f} d_0}{h^2} \right) + (H X \dot{R}) \cdot (\dot{R}_0) \dot{g} \right] &= \\ \frac{1}{2 \mu h v^2} \left[\frac{-\dot{f} d_0}{h^2} h^2 (\dot{R} \cdot R_0) + \dot{g} h^2 \dot{f} \right] &= \\ \frac{h \dot{f}}{2 \mu v^2} \left[\dot{g} - \frac{d_0}{h^2} (\dot{g} d - g v^2) \right] &= \\ \frac{h \dot{f}}{2 \mu v^2} \left[\dot{g} + \frac{d_0}{h} \left(\frac{\mu}{r} g - d_0 \right) \right] \end{aligned}$$

The second part of W_3 comes from differentiating \dot{f} and \dot{g} directly with respect to $\mu(t_0)$.

$$\dot{f} = - \frac{\sqrt{\mu}}{r r_0} \beta F_3 = - \frac{\sqrt{\mu a}}{r r_0} \sin \theta$$

$$\dot{g} = 1 - \frac{1}{r} \beta^2 F_2 = 1 - \frac{a}{r} (1 - \cos \theta)$$

$$= \frac{1}{r} (r_0 \cos \theta + \frac{d_0}{\sqrt{\mu}} \sqrt{a} \sin \theta)$$

$$\frac{\partial \dot{f}}{\partial \mu(t_0)} = + \frac{1}{2} \frac{\dot{f}}{\mu} - \frac{\dot{f}}{r} \frac{\partial r}{\partial \mu(t_0)} = \frac{\sqrt{\mu a}}{r r_0} \cos \theta \frac{\partial \theta}{\partial \mu(t_0)}$$

Using the third expression for \dot{g}

$$\frac{\partial \dot{g}}{\partial \mu(t_0)} = - \frac{\dot{g}}{r} \frac{\partial r}{\partial \mu(t_0)} - \frac{1}{2r} \frac{d_0}{\mu^{3/2}} \beta F_3 + \frac{1}{r} \left(-r_0 \sin \theta + \frac{d_0}{\sqrt{\mu}} \sqrt{a} \cos \theta \right) \frac{\partial \theta}{\partial \mu(t_0)}$$

$$\text{So } \frac{h}{v^2} \left(\frac{\partial \dot{g}}{\partial \mu(t_0)} - \dot{g} \frac{\partial \dot{f}}{\partial \mu(t_0)} \right)$$

$$= \frac{h}{v^2} \left[- \frac{\dot{f} \dot{g}}{2 \mu} + \frac{d_0 \dot{f}^2 r_0}{2 \mu^2} \right.$$

$$\left. + \frac{\sqrt{\mu a}}{r^2 r_0} \left(r_0 \cos^2 \theta + \frac{d_0}{\sqrt{\mu}} \sqrt{a} \sin \theta \cos \theta + r_0 \sin^2 \theta - \frac{d_0}{\sqrt{\mu}} \sqrt{a} \sin \theta \cos \theta \right) \frac{\partial \theta}{\partial \mu(t_0)} \right]$$

$$\begin{aligned}
& \frac{h}{v^2} \left[-\frac{\dot{f} \dot{g}}{2\mu} + \frac{d_0 \dot{f}^2 r_0}{2\mu^2} + \frac{\mu a}{r^2} \frac{\partial \theta}{\partial \mu(t_0)} \right] \\
&= \frac{h \dot{f}}{2\mu v^2} \left[-\dot{g} + \frac{r_0 d_0 \dot{f}}{\mu} \right] + \frac{h}{2v^2 r^3} \left(\Delta t + \frac{d_0}{\mu} \beta^2 F_2 \right)
\end{aligned}$$

Finally

$$\begin{aligned}
W_3 &= \frac{h}{2v^2 r^3} \left(\Delta t + \frac{d_0}{\mu} \beta^2 F_2 \right) \\
&+ \frac{\dot{f} d_0}{2\mu v^2 h} \left(\frac{\mu}{r} g - d_0 + \frac{\dot{f} r_0 h^2}{\mu} \right)
\end{aligned}$$